Kibernetika, № 5, 1969.
2. Krasovskii, N. N. and Subbotin, A. I., Position Differential Games. Moscow, "Nauka", 1974.
3. Osipov, Iu. S., A differential guidance game for systems with aftereffect. PMM Vol. 35, № 1, 1971.
4. Krasovskii, N. N. and Osipov, Iu. S., Linear differential-difference games. Dokl. Akad. Nauk SSSR, Vol. 197, № 4, 1971.
5. Kriazhimskii, A. V. and Osipov, Iu. S., Differential- difference game of encounter with a functional target set. PMM Vol. 37, № 1, 1973.
6. Osipov, Iu. S., Alternative in a differential-difference game. Dokl. Akad. Nauk SSSR, Vol. 197, № 5, 1971.
7. Tarlinskii, S. I., On a linear differential game of encounter. PMM Vol. 37, № 1, 1973.
8. Dunford, N. and Schwartz, J. T., Linear Operators, Part I. General Theory. New York, Interscience Publishers, Inc., 1958.
9. Fan' Chi, Minimax theorems. In: Infinite Antagonistic Games. Moscow, Fizmatgiz, 1963.

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ON THE PRECESSION EQUATIONS OF GYROSCOPIC SYSTEMS

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The analysis of equations of gyroscopic systems almost always necessitates the separation of fast nutation motions from those of slow precession. Simplified equations for the determination of these two kinds of motion can be obtained by various means [1]. The method of fractional analysis [2] used here for formalizing the passing to precession equations is based on the combination of methods of the theory of similarity and dimensionality with asymptotic methods of the theory of differential equations. The asymptotic behavior of solutions of complete equations of the gyroscopic system motions is investigated in the case in which the ratio of characteristic times T_n and T_p of nutation and precession motion components tends to zero.

Definition. Equations whose solutions for the slow components of motion represent for times of order T_p the zero order approximation with respect to the small parameter $\mu = T_n / T_p$, where T_n and T_p are the characteristic times of nutation and precession components, respectively, are called precession equations of gyroscopic systems. The exact meaning of this definition is made clear subsequently.

Let us briefly consider the problem of passing to precession equations. Some of such problems were considered earlier in [3, 4].

The general equations of gyroscopic systems are of the form [1]:

$$q^{\cdot} = A^{-1}p, \quad p^{\cdot} = \frac{1}{2} p^T (A^{-1})^T \frac{\partial A}{\partial q} A^{-1}p + Q + (B + H) A^{-1}p \qquad (1)$$
$$q(0) = q^{\circ}, \quad p(0) = p^{\circ}$$

where the dot denotes differentiation with respect to time; $q = (q_j)$ is the vector of generalized coordinates that define the angular position of elements of the gyroscopic system; $A = (A_{jk})$ is the matrix of moments of inertia; $B = (B_{jk})$ is the matrix of dissipation coefficients; $H = (H_{jk})$ is the matrix of gyroscopic coefficients; $Q = (Q_j)$ is the vector of generalized forces acting on the system; q° and p° are initial values of the generalized coordinates and momenta, respectively, and T is the symbol of transposition. Further constructions are derived on certain assumptions about the behavior of functions A_{jk}, B_{jk}, H_{jk} and Q_j in Eq. (1). For the sake of brevity, some of the requirements, such as those about the smoothness of functions, uniform boundedness, etc. are not explicitly stated. It is assumed that the considered functions satisfy all necessary conditions.

Let us normalize the equations of motion (1) of the gyroscopic system [2, 5] and set

$$q = q_* y, \quad A^{-1}p = \omega_* z, \quad A = A_* a, \quad B = B_* b, \quad H = H_* h, \qquad (2)$$
$$Q = Q_* f$$

where the asterisk denotes characteristic values of parameters of the considered class of motions. These are chosen so that the elements of dimensionless matrices y, z, a, b, h and f are of the order of unity. We select the time constant T_*, characteristic for this kind of gyroscopic system motion, as the unit of time, and introduce the related dimensionless time.

We further assume that the geometric and mass characteristics of elements of the gyroscopic system are, respectively, quantities of the same order, and set

$$A_* = \max \{A_{jk}\}, \quad B_* = \max \{B_{jk}\}, \quad H_* = \max \{H_{jk}\}$$

The quantity Q_* in (2) represents the maximum value of the generalized forces in the considered region of variation of time and generalized coordinates. This region is selected on the basis of the specific construction of the gyroscopic system and its operation conditions.

The angles of turn of structural components of the gyroscopic system are usually taken as the generalized coordinates. We assume these angles can be fairly great so that $q_* = 1$. Angular velocities of the gyroscopic system are determined by the magnitude and frequencies of control signals and perturbations. For example, the characteristic values ω_* of a uniaxial gyrostabilizer are determined by parameters of the feed-back circuit.

It is assumed here that the gyroscopic system control is not "rigid" and that the "fast" and "slow" angular velocity components are of comparable order of magnitude. It was shown in [5] that the fast component of angular velocity is in the form of oscillations whose amplitude is equal to the product of a quantity of the order of a very small fraction of a radian by the nutation frequency, while the slow component is determined by the gyroscope precession induced by the applied moments. We select ω_* equal to the characteristic angular velocity of precession $\omega_* = Q_* / H_*$. The substitution of selected characteristic parameters (2) into (1) yields the system of equations

$$\begin{aligned}
&dy/dt = \frac{T_*}{T_p} z, \quad y(0) = \frac{q^\circ}{q_*} \equiv y^\circ \\
&\frac{T_n}{T_*} \frac{d}{dt}(az) = \frac{1}{2} \frac{T_n}{T_p} z^T \frac{\partial a}{\partial y} z + f - (\varkappa b + h) z \\
&z(0) = A^{-1} p^\circ / \omega_* \equiv z^\circ \\
&(T_p = H_* / Q_*, \quad T_n = A_* / Q_*, \quad \varkappa = B_* / H_*)
\end{aligned} \tag{3}$$

Parameters T_n and T_p are called the nutation and precession time constants, respectively. We assume that $T_n \ll T_p$ (which is valid for the majority of gyroscopic systems).

If the considered system is nonautonomous, we must introduce the characteristic perturbation time T_f whose magnitude in the case of harmonic perturbations can be of the order of the perturbation period. If $T_f \sim T_n$, the system is considered to be subjected to fast (high-frequency) perturbations, while for $T_f \sim T_p$ the latter are considered to be slow. Generally it is possible to set in (3)

$$\varkappa = \mu^\alpha, \quad \mu = T_n / T_p, \quad 0 \leqslant \alpha \leqslant 1 \tag{4}$$

At the limit $\alpha = 0$ $(\varkappa = 1)$ we have a strongly damped gyroscopic system, an example of which is provided by the float-type gyroscope. The limit case of $\alpha = 1$ $(\varkappa = \mu)$ corresponds to a weakly damped gyroscopic system such as, for example, one with low friction in gimbal axes. Formulas of the kind of (4) were used in [6] for analyzing periodic solutions of singularly perturbed equations arising from gyroscopic systems for $0 < \alpha < 1$. We restrict our analysis to the following case with different parameters $\varkappa$ and T_f.

Case A. The perturbations are slow $(T_f = T_p)$ and the gyroscopes of the system are strongly damped, i.e. $\varkappa = 1$.

As the characteristic time in (3) we take $T_* = T_p$, then

$$\begin{aligned}
&dy/dt = z \\
&\mu \frac{d}{dt}[a(y,t) z] = \frac{\mu}{2} z^T \frac{\partial a(y,t)}{\partial y} z + f(y,t) - [\varkappa b(y,t) + h(y,t)] z \\
&y(0) = y^\circ, \quad z(0) = z^\circ, \quad \varkappa = 1
\end{aligned} \tag{5}$$

The above are differential equations with a small parameter at derivatives. The confluent system of equations for system (5) is of the form [7]

$$\begin{aligned}
&dv/dt = w, \quad v(0) = y^\circ \\
&f(v,t) - [b(v,t) + h(v,t)] w = 0
\end{aligned} \tag{6}$$

The sufficient conditions of applicability of Eqs. (6) for defining the motion of a gyroscopic system are provided by the following theorem.

Theorem 1. Let us assume that the following conditions are satisfied:

a) The determinant of matrix $b(y,t) + r(y,t)$ within the region $D = \{t, y : t \in [0, c_1], \| y \| \leqslant y_1\}$ of variation of variables is nonzero.

b) The solution $v = v(t)$ of system (6) exists and is unique for all $t \in [0, c_1]$.

c) For all t and y from some bounded closed region that contains the confluent solution $v = v(t)$ there exists an asymptotic stability of the equilibrium point

$w(y, t) = [b(y, t) + h(y, t)]^{-1} f(y, t)$ of the adjoint system of differential equations

$$\frac{d}{d\tau}[a(y, t) z] = f(y, t) - [b(y, t) + h(y, t)] z \tag{7}$$

in which y and t are parameters and $\tau = t / \mu$ is the fast time.

Then $v(t)$ and $w(t)$ represent the zero order asymptotic approximation of the exact solution $y(t, \mu)$ and $z(t, \mu)$ of complete Eqs. (5), i.e.

$$\| y(t, \mu) - v(t) \| = O(\mu) \quad \text{for} \quad t = O(1) \tag{8}$$

Formula (8) explicitly implies the existence of constants c_1 and c_2 independent of μ such that the estimate

$$\| y(t, \mu) - v(t) \| \leqslant c_2 \mu$$

is valid for all values of slow time t in the interval $0 \leqslant t \leqslant c_1$. Constant c_1 can be arbitrary, but its selection evidently affects c_2 and μ_0.

The proof of Theorem 1 follows directly from Tikhonov's theorem [7], if one considers that in this case the influence region of the stable root of the confluent equation is unbounded owing to the linearity of the adjoint system (7) of equations in z [7]. Only such z that do not violate normalization conditions (2) are used for solving specific problems. Note that solutions of the complete and confluent systems in fast variables z are close to each other outside some initial time interval defined by a quantity of order $\mu \ln (1 / \mu)$. It is called the exponential boundary layer in time.

Thus according to the above definition formulas (6) represent precession equations of gyroscopic systems. When passing to limit from the complete to confluent equations the proposed method of derivation of precession equations yields the same equations as those obtained in [1].

Case B. Perturbations are slow $(T_f = T_p)$ and the system's gyroscopes are weakly damped, i.e. $\varkappa = \mu$. (Conservative systems entirely free of damping belong to this class). In this case the conditions of Tikhonov's theorem are not satisfied when passing to limit from system (5) to system (6) for $\mu \to 0$.

The passing to precession equations in Case B can be validated by using, for example, the device of averaging in the form proposed by Volosov [8].

Let us substitute $t = \mu\tau$ into (5). For $\varkappa = \mu$ the equations of motion of the gyroscopic system are of the form

$$\frac{dy}{d\tau} = \mu z, \quad \frac{dt}{d\tau} = \mu \tag{9}$$

$$\frac{d}{d\tau}[a(y, t) z] = \frac{\mu}{2} z^T \frac{\partial a(y, t)}{\partial y} z + f(y, t) - [\mu b(y, t) + h(y, t)] z$$

$$y(0) = y^\circ, \quad z(0) = z^\circ$$

For the fast time the form of system (5) corresponds to the selection of the nutation time constant $T_* = T_n$ as the unit of time in (3). An asymptotically great interval $0 \leqslant \tau \leqslant c_1 / \mu$ of variation of the fast time τ corresponds to the interval $0 \leqslant t \leqslant c_1$ of variation of the slow time t.

Setting in (9) $\mu = 0$, we obtain the so-called generating system of differential equations

$$d/d\tau\, [a(v, t) w] = f(v, t) - h(v, t) w, \quad w(0) = z^\circ \tag{10}$$

where v and t are parameters and, consequently, a_{jk}, h_{jk} and f_j are constant quantities with respect to τ. The generating system defines similarly to adjoint Eqs.(7) the fast motions of the considered mechanical system at every instant of time.

Let us consider the following variants of Case B:

B1) $\det h \neq 0$ and B2) $\det h = 0$.

For $\det h \neq 0$ the solution of system (10) is represented by the sum of the constant component $h^{-1}f$ and of the set of harmonic components that are nutational oscillations of the system [1].

The equation which determines the approximate solution of system (9) in terms of slow variables to within the first order of smallness with respect to μ is of the form

$$\frac{dv}{d\tau} = \mu \langle w \rangle, \qquad \langle w \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T w(z^\circ, v, t, \tau)\, d\tau \tag{11}$$

where $\langle w \rangle$ denotes the result of averaging the solution of the generating system (10).

Since the oscillating components of the solution vanish in computation, (11) assumes the form

$$dv/dt = h^{-1}(v, t)\, f(v, t), \quad v(0) = y^\circ \tag{12}$$

In this case Eqs. (12) are precession equations of the gyroscopic systems. Note that Eqs. (12) differ from the simplified equations for such systems derived by equating to zero some parts of the expressions for kinetic energy [1]. The precession equations obtained here are simpler than the latter, since they do not contain dissipation terms.

When the number of the system generalized coordinates is odd, we have Case B2 with $\det h(y, t) = 0$, and the result of averaging the solution of the generating system (if the limit of $\langle w \rangle$ exists) depends on initial conditions for the fast variables z. Hence in accordance with the general averaging method [8] the transition to precession equations reduces in this case to the derivation of a substitution which would increase the number of coordinates of the vector of slow variables.

As an example, we present in explicit form the equation for the slow components of the solution of system (9) for $\det h = 0$ in the particular case when $a(y, t)$ is a unit matrix. Let us assume that the dimension of vector z is $2r+1$, and that matrix h is independent of y and has one zero eigenvalue, while the remaining eigenvalues of the skew symmetric matrix h are purely imaginary. Let furthermore $f(y, t) = 0$. (This is a meaningful requirement, since for $f(y, t) \neq 0$ and $\det h = 0$ a limit for the solution of the generating system may not exist).

On these assumptions system (9) assumes the form

$$\frac{dy}{d\tau} = \mu z, \qquad \frac{dt}{d\tau} = \mu, \qquad \frac{dz}{d\tau} = -[h(t) + \mu b(y, t)]\, z \tag{13}$$

$$y(0) = y^\circ, \quad z(0) = z^\circ$$

and the generating system

$$dw/d\tau = -h(t)\, w, \quad w(0) = z^\circ \tag{14}$$

Using in (13) the vector of arbitrary constants obtained by solving the generating system (14) as the new variables, we can obtain a system of standard form equations that is equivalent to (13) [8]. Carrying out the averaging procedure in that system for the augmented vector of slow variables we obtain the following equations of $(2r+2)$ order:

$$\frac{dv}{dt} = k\zeta(t) \quad (k = \zeta^T(t)\, z) \tag{15}$$

$$\frac{dk}{dt} = -k\zeta^T(t)\left(b(v,t)\zeta(t) + \frac{d\zeta(t)}{dt}\right)$$
$$v(0) = y^\circ, \quad k(0) = \zeta^T(0)z^\circ, \quad \|\zeta(t)\| = 1$$

where $\zeta(t)$ is the normalized eigenvector of matrix $h(t)$ which corresponds to the zero eigenvalue, k is the new scalar variable which is an element of the vector of arbitrary constants related to the particular solution with zero eigenvalue. Note that k is an integral of the generating system (14) and a projection of vector z on the direction determined by the eigenvector $\zeta(t)$. Equations (15) can be considered as the precession equations for the gyroscopic system (13).

The following theorem follows directly from the theorem about the first approximation of the general method of averaging.

Theorem 2. In the time interval $0 \leqslant \tau \leqslant c_1/\mu$ and for fairly small μ solutions of Eqs.(9) and (12) (Case B1) and of Eqs.(13) and (15) (Case B2) can be as close as desired.

Note that generally such closeness does not exist with respect to fast variables.

A strongly damped system subjected to fast perturbations can be analyzed similarly to Case B by using the general method of averaging.

Note that the presence in a gyroscopic system of fast perturbations accompanied by weak damping may lead to the appearance of resonance effects, which necessitates the use of corresponding results of the theory of the averaging method [8].

The above investigation shows that the equations, whose solution $v(t)$ within the problem precession times, i.e. $\|y(t,\mu) - v(t)\| = O(\mu)$ for $t = O(1)$, lies in the μ-neighborhood of the exact solution of the gyroscopic system (5) for the slow variables $y(t,\mu)$, can be taken as the precession equations of such systems. If in the last formula μ is directed to zero, the solution of the precession equations $v(t) = \lim_{\mu\to 0} y(t,\mu)$ obtained in this manner for $0 \leqslant t \leqslant c_1$, i.e. $v(t)$, is unique. We would point out, however, that $v(t)$ does not uniquely determine precession equations, unlike a power series that simultaneously represents asymptotically a particular function and an infinite set of functions. Hence the retention in approximate equations, for example, of terms of order μ^2 is not a fundamental condition for obtaining the approximate solution of equations of a gyroscopic system with the specified above accuracy. It is from this point of view that the question of neglecting dissipative terms in precession equations in Case B is to be decided.

The question of validity of precession equations for infinite time intervals requires separate consideration, since the conditions of the above two theorems ensure the closeness of the complete and precession equations only in a finite interval of the slow time t.

It is not difficult to see that it is not generally possible to approximate the solution of complete equations for $t \to \infty$ by solutions of precession equations without imposing further restrictions. Such closeness is present when the solutions of precession equations are stable for constantly acting perturbations.

Such result was obtained in (9) for a linear system in Case A. In the nonlinear case it is possible to use the theorems [10] which ensure the closeness of the complete and confluent equations in an infinite time interval when the solutions of confluent equations are stable in the first approximation (*) (See Foot-note at the next page).

The results of [8, 11] can be used for proving the validity of passing to precession equations when using the general scheme of the averaging method for an infinite time interval. From the theorem formulated in [11] follows Theorem 3.

Theorem 3. Let the precession equations (12) have the trivial solution $v = 0$, and let there exist a positive definite scalar function $V(t, v)$ whose derivative in virtue of the precession equation (12) is the negative definite function

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial v}\langle w\rangle \leqslant -W_1(v), \quad W_2(v) \leqslant V(t, v) \leqslant W_3(v)$$

where $W_1(v)$, $W_2(v)$ and $W_3(v)$ are positive definite functions. Then for fairly minor initial conditions for the slow variables y and fairly small μ the solutions of complete and precession equations are close to each other in an infinite time interval.

Thus the theory of singularly perturbed equations makes it possible to reduce the problem of passing in a gyroscopic system to precession equations to the problem of separating fast and slow motions. Depending on the system behavior when the small parameter tends to zero the results of Tikhonov [7] and Vasil'eva [12], or the general method of averaging [8] can be used.

The described methods do not, evidently, represent all methods of investigation of singularly perturbed systems, which can be applied for passing to precession equations of the theory of gyroscopes [13—15]. In particular, the scheme devised in [16] which does not require an explicit solution of the generating system can be used. The considerable possibilities of the regulatization method of passing to a space of considerable dimensions, proposed by Lomov [17, 18], should be mentioned.

REFERENCES

1. Merkin, D. R., Gyroscopic Systems. Gostekhizdat, Moscow, 1956.
2. Kline, S. J., Similitude and Approximation Theory. McGraw-Hill, New York, 1965.
3. Novozhilov, I. V., On lowering the order of equations of gyroscopic systems. Inzh. Zh. MTT, № 5, 1966.
4. Novozhilov, I. V., On the use of asymptotic expansions of the theory of differential equations with a small parameter at the higher derivative for investigating gyroscopic systems. Izv. Akad. Nauk SSSR, MTT, № 4, 1970.
5. Novozhilov, I. V., Approximate methods of gyroscopic system investigation. Book: Development of Gyroscopic and Inertial System Mechanics, "Nauka", Moscow, 1973.
6. Balachandra, M., Periodic solutions of singularly perturbed equations arising from gyroscopic systems. SIAM J. Appl. Math., Vol. 24, № 4, 1973.
7. Tikhonov, A. N., Systems of differential equations with small parameters at derivatives. Matem. Sb., Vol. 31 (73), № 3, 1952.
8. Volosov, V. M. and Morgunov, B. I., The Method of Averaging in the Theory of Nonlinear Oscillating Systems. Izd. MGU, Moscow, 1971.
9. Novozhilov, I. V., On passing to precession equations for gyroscopic systems in an infinite time interval. Izv. Akad. Nauk SSSR, MTT, № 5, 1971.

*) A. I. Kobrin and Iu. G. Martynenko, On the question of passing to limit in systems of differential equations with a small parameter at derivatives in an infinite time interval. Nauchn. Tr. Inst. Mekhaniki MGU, Moscow, 1970.

10. Klimushev, A. I. and Krasovskii, N. N., Uniform asymptotic stability of systems of differential equations with a small parameter in the derivative terms. PMM Vol. 25, № 4, 1961.
11. Aleshin, B. V., On the applicability of the averaging method in an unbounded time interval. Vest. MGU, Moscow, Ser. Matem. Mekhan., № 6, 1970.
12. Vasil'eva, A. B. and Butuzov, V. F., Asymptotic Expansions of Solutions of Singularly Perturbed Equations. "Nauka", Moscow, 1973.
13. Martynenko, Iu. G., Asymptotic methods in the theory of gyroscopic devices. Nauch. Tr. Inst. Mekhaniki MGU, Moscow, № 29, 1973.
14. Kobrin, A. I., On the evaluation of the precision of precession equations of gyroscopic stabilizers. Nauch. Tr. Inst. Mekhaniki MGU, Moscow, № 29, 1973.
15. Sethna, P. R. and Balachandra, M., Transients in high-spin gyroscopic systems. Trans. ASME, Ser. E, J. Appl. Mech., Vol. E41, № 3, 1974.
16. Aleshin, B. V., The asymptotic behavior of a particular system of differential equations. Dokl. Akad. Nauk SSSR, Vol. 210, № 2, 1973.
17. Lomov, S. A., The formalism of nonclassical theory of perturbations. Dokl. Akad. Nauk SSSR, Vol. 212, № 1, 1973.
18. Kobrin, A. I. and Martynenko, Iu. G., On a method of derivation of an asymptotic solution of the problem of motion of a gyroscope suspended in gimbals. Izv. Akad. Nauk SSSR, MTT, № 3, 1971.

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THE SECOND LIAPUNOV METHOD IN THE THEORY OF PHASE SYNCHRONIZATION

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A certain analog of the Liapunov second method is constructed for dynamic systems with cylindrical phase space. The known results obtained by it for second order dynamic systems are extended to systems with cylindrical phase space of arbitrary dimensions. The derived theorems are used for analyzing the operation of a system of two synchronous machines and for investigating the automatic phase frequency as a "whole".

The working modes of systems of automatic phase frequency control (APFC) are usually such that the phase difference $\sigma(t)$ between the reference generator that is being synchronized is a bounded function of time $t \in (0, +\infty)$. It is often possible to establish on the basis of boundedness of $\sigma(t)$ that for $t \to +\infty$ there exists a finite limit of $\sigma(t)$ for autononmous APFC systems [1, 2]. The presence of such limit means that the considered working mode of the APFC is one of capture [1]. Similar statements are also valid for working modes of synchronous motors, except that then the phase difference between the rotating magnetic field and the rotor is represented by function $\sigma(t)$ [3—6].

A certain analog of the Liapunov second method is derived below, which makes it possible to obtain effective sufficient conditions of boundedness or unboundedness for function